Exercise 1

Given the vectors:

$$\bar{a} = 3\hat{x} + 2\hat{y} + 5\hat{z} \qquad \bar{b} = 6\hat{x} + \hat{y} - 4\hat{z}$$

evaluate their magnitudes and $\bar{a} + \bar{b}$, $\bar{a} \cdot \bar{b}$, and $\bar{a} \times \bar{b}$.

The magnitudes are:

$$a = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38} \qquad b = \sqrt{6^2 + 1^2 + (-4)^2} = \sqrt{53}$$

The sum of two vectors can be obtained by summing the corresponding components, therefore:

$$\bar{a} + \bar{b} = (3\hat{x} + 2\hat{y} + 5\hat{z}) + (6\hat{x} + \hat{y} - 4\hat{z}) =$$
$$= (3 + 6)\,\hat{x} + (2 + 1)\,\hat{y} + (5 - 4)\,\hat{z} = 9\hat{x} + 3\hat{y} + \hat{z}$$

Exercise 1 (cont.)

The dot product is a scalar given by the sum of the products of the components, that is:

$$\bar{a} \cdot \bar{b} = (3\hat{x} + 2\hat{y} + 5\hat{z}) \cdot (6\hat{x} + \hat{y} - 4\hat{z}) =$$
$$= 3 \cdot 6 + 2 \cdot 1 + 5 \cdot (-4) = 18 + 2 - 20 = 0$$

since the dot product is null, the vectors are orthogonal.

The cross product is given by the determinant:

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 2 & 5 \\ 6 & 1 & -4 \end{vmatrix} = [2 \cdot (-4) - 5 \cdot 1]\,\hat{x} + [5 \cdot 6 - 3 \cdot (-4)]\,\hat{y} +$$
$$+ (3 \cdot 1 - 2 \cdot 6)\,\hat{z} = -13\hat{x} + 42\hat{y} - 9\hat{z}$$

Exercise 2

Consider a rectangular waveguide with dimensions $a = 2$ cm and $b = 1$ cm. The guide is working in the fundamental mode. Two signals at $f_1 = 10$ GHz and $f_2 = 10.06$ GHz, respectively, are injected in the waveguide at $z = 0$. Evaluate the time delay between the two signals at $z = l = 1$ m.

The time needed to travel a space l inside the waveguide is:

$$t = \frac{l}{v_g}$$

where the group velocity v_g is given by:

$$v_g = c\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

with f_c the cutoff frequency of the mode.

Exercise 2 (cont.)

The fundamental mode is the TE_{01}. Its cutoff frequency depends only on a and is:

$$f_c = \frac{c}{2a} = \frac{3 \cdot 10^8}{2 \cdot 2 \cdot 10^{-2}} = \frac{30}{4} 10^9 = 7.5 \text{ GHz}$$

Given f_c we have:

$$v_{g_1} = c\sqrt{1 - \left(\frac{7.5}{10}\right)^2} \simeq 1.9843 \cdot 10^8 \text{ m/s}$$

$$v_{g_2} = c\sqrt{1 - \left(\frac{7.5}{10.06}\right)^2} \simeq 1.9994 \cdot 10^8 \text{ m/s}$$

and the time delay is:

$$\Delta t = t_1 - t_2 = \frac{l}{v_{g_1}} - \frac{l}{v_{g_2}} = \frac{1}{1.9843 \cdot 10^8} - \frac{1}{1.9994 \cdot 10^8} \simeq$$

$$\simeq 3.81 \cdot 10^{-11} \text{ s} = 38.1 \text{ ps}$$

Note that the higher the frequency, the faster the signal.

Exercise 3

A uniform plane wave propagating in vacuum along the z–direction has real–valued electric field components:

$$E_x = \cos(\omega t - kz), E_y = 2\sin(\omega t - kz).$$

Its phasor form has the form $\bar{E} = (A\hat{x} + B\hat{y})e^{\pm jkz}$.

Determine the numerical values of the complex–valued coefficients A, B and the correct sign of the exponent.

Determine the polarization of this wave.

Solution

First of all, we need to manipulate the real-valued expression in order to obtain an expression with phasor components:

From $E_x(z, t) = \cos(\omega t - kz)$ we get immediately

$$E_x(z) = A\mathrm{e}^{-jkz}$$

with $A = 1$.

Exercise 3 (cont.)

For the second expression we have that:

$$\Re\left\{E_y(z)e^{j\omega t}\right\} = \Re\left\{Be^{-jkz}e^{j\omega t}\right\} = 2\sin(\omega t - kz)$$

that is

$$\Re\{B\left[\cos(\omega t - kz) + j\sin(\omega t - kz)\right]\} = 2\sin(\omega t - kz)$$

and this implies $B = -2j$, therefore

$$E_y(z) = -j2e^{-jkz}$$

and

$$\bar{E} = (\hat{x} - j2\hat{y})e^{-jkz}$$

The polarization of the wave is elliptical because the x and y components are in quadrature and amplitudes are different.

Exercise 4

Consider the electromagnetic field given by

$$\bar{E} = [\hat{x} + E_y\hat{y} + (2 + j5)\hat{z}]\, e^{-j2.3(-0.6x+0.8y)}$$
$$\bar{H} = (H_x\hat{x} + H_y\hat{y} + H_z\hat{z})\, e^{-j2.3(-0.6x+0.8y)}$$

where E_y, H_x, H_y, H_z are all independent of x, y, and z. Assume that the field is propagating in vacuum $(\mu = \mu_0$ and $\varepsilon = \varepsilon_0)$ and determine:

- The component E_y in order to have an uniform plane wave;
- The components H_x, H_y, H_z;
- The frequency and corresponding wavelength;
- The state of polarization of the wave.

Exercise 4 (cont.)

Solution:

By looking at the general form of a plane wave, it should be clear that:

$$2.3(-0.6x + 0.8y) = \bar{k} \cdot \bar{r} = \bar{k} \cdot (x\hat{x} + y\hat{y} + z\hat{z})$$

furthermore, since propagation is in vacuum, then

$$k = \beta - j\alpha = \beta$$

therefore:

$$\bar{k} = \beta\hat{k} = 2.3(-0.6\hat{x} + 0.8\hat{y})$$

and since $(-0.6\hat{x} + 0.8\hat{y})$ is a vector of unitary magnitude, then $\beta = 2.3$;

We are dealing with a uniform plane wave in sinusoidal state. In this case

$$\nabla \cdot \bar{E} = 0$$

implies

$$\hat{k} \cdot \bar{E} = 0$$

Exercise 4 (cont.)

and it follows that

$$[\hat{x} + E_y \hat{y} + (2 + j5)\hat{z}] \cdot (-0.6\hat{x} + 0.8\hat{y}) = -0.6 + 0.8E_y = 0$$

then

$$E_y = \frac{0.6}{0.8} = 0.75$$

and

$$\bar{E} = [\hat{x} + 0.75\hat{y} + (2 + j5)\hat{z}]\, e^{-j2.3(-0.6x+0.8y)}$$

We know, from the relations between $\bar{E}$and $\bar{H}$, that:

$$\bar{H} = \frac{1}{\eta}\hat{k} \times \bar{E}.$$

Exercise 4 (cont.)

In a vacuum $\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377 \simeq 120\pi\,\Omega$, while

$$\hat{k} \times \bar{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -0.6 & 0.8 & 0 \\ 1 & 0.75 & 2+j5 \end{vmatrix},$$

so that

$$H_x = \frac{0.8(2+j5)}{377} \simeq (4.24 + j10.6) \cdot 10^{-3}$$
$$H_y = \frac{0.6(2+j5)}{377} \simeq (3.18 + j7.95) \cdot 10^{-3}$$
$$H_z = -\frac{0.6 \cdot 0.75 + 0.8}{377} \simeq 3.31 \cdot 10^{-3}$$

Exercise 4 (cont.)

The wavelength is determined as

$$\lambda = \frac{2\pi}{\beta} \simeq 2.73 \text{ m}$$

and the frequency as

$$f = \frac{c}{\lambda} = \frac{\beta c}{2\pi} \simeq 109.8 \cdot 10^6 \text{ Hz} = 109.8 \text{ MHz}$$

Separating the real and imaginary parts of the magnitude E of $\bar{E}$, we have:

$$\Re\{E\} = E_R = \hat{x} + 0.75\hat{y} + 2\hat{z}$$
$$\Im\{E\} = E_I = 5\hat{z}$$

Clearly, E_R and E_I are neither collinear nor mutually orthogonal in space. Therefore the polarization of the wave cannot possibly be linear nor circular; it must be elliptical.

Exercise 5

Design a rectangular waveguide which, at $f = 10$ GHz, will operate in the TE_{10} mode with 25 percent safety factor ($f \geq 1.25 f_c$) when the interior of the guide is filled with air. It is required that the mode with the next higher cutoff will operate at 25 percent below its cutoff frequency.

Solution:

For the TE_{10} mode,

$$f_{10} = \frac{c}{2a}$$

Therefore the lower bound for the dimension a is

$$a = \frac{c}{2f} = \frac{3 \cdot 10^8}{2 \cdot 10 \cdot 10^9} = 1.5 \cdot 10^{-2} = 1.5 \text{ cm}$$

The upper bound is found from the condition $f \geq 1.25 f_c$.

Thus

$$a \leq 1.25 \frac{c}{2f} = 1.25 \frac{3 \cdot 10^8}{2 \cdot 10 \cdot 10^9} = 1.875 \cdot 10^{-2} = 1.875 \text{ cm}$$

so that a must be chosen such that $1.5 \leq a \leq 1.875$.

Exercise 5 (cont.)

The cutoff of the TE_{20} is:

$$f_{20} = \frac{c}{a}$$

therefore, in the worst case:

$$0.75 f_{20} = 0.75 \frac{c}{1.875 \cdot 10^{-2}} = 12 \text{ GHz}$$

and this is beyond the operating frequency.

The other wave with the next higher cutoff frequency is the TE_{01} mode. Its cutoff frequency is

$$f_{01} = \frac{c}{2b}$$

hence we must have:

$$b \leq 0.75 \frac{c}{2f} = 0.75 \frac{3 \cdot 10^{8}}{2 \cdot 10 \cdot 10^{9}} = 1.125 \cdot 10^{-2} = 1.125 \text{ cm}$$

Therefore the choice $a = 1.875$ cm, $b = 1.125$ cm can satisfy the requirements.

Exercise 6

A source located at $z = 0$ generates an electromagnetic pulse of duration of 10^{-5} seconds, given by:
$\bar{E}(0,t) = \hat{x}E_0\left[u(t) - u(t-10^{-5})\right]$, where $u(t)$ is the unit step function and E_0 is a constant. The pulse is launched towards the positive z–direction. Determine expressions for $\bar{E}(z,t)$ and $\bar{H}(z,t)$.

Solution

For a forward–moving wave, we have
$\bar{E}(z,t) = \bar{F}(z-ct) = \bar{F}(0-c(t-z/c))$, which implies that $\bar{E}(z,t)$ is completely determined by $\bar{E}(z,0)$ or, alternatively, by $\bar{E}(0,t)$:

$$\bar{E}(z,t) = \bar{E}(z-ct,0) = \bar{E}(0,t-z/c)$$

Using this property , we find for the electric and magnetic fields:

$$\bar{E}(z,t) = \bar{E}(0,t-z/c) = \hat{x}E_0\left[u(t-z/c) - u(t-z/c-10^{-5})\right]$$

$$\bar{H}(z,t) = \frac{1}{\eta}\hat{z}\times\bar{E}(z,t) = \hat{y}\frac{E_0}{\eta}\left[u(t-z/c) - u(t-z/c-10^{-5})\right]$$

Exercise 7

Suppose a uniform plane wave in empty space has the electric field:

$$\bar{E}(z) = \hat{x}1000e^{-j\beta_0 z} \text{ V/m}$$

its frequency being $f = 20$ MHz.

- What is its direction of travel? Its amplitude? Its vector direction in space?
- Find the associated $\bar{H}$ field.
- Express $\bar{E}$ and $\bar{H}$ in real-time form.
- Find the wavenumber β_0, the phase velocity, and the wavelength of this electromagnetic wave.

Solution:

The minus sign in the exponential term reveals a positive z traveling wave.

The real amplitude is $E_{0+} = 1000$ V/m with the vector field $\hat{x}$ directed in space.

Exercise 7 (cont.)

Using $\hat{z} \times \bar{E} = \eta_0 \bar{H}$ gives the magnetic field:

$$\bar{H}(z) = \hat{y}\frac{1000}{120\pi}e^{-j\beta_0 z} \simeq \hat{y}2.65e^{-j\beta_0 z} \text{ A/m}$$

The real-time fields are obtained from:

$$\bar{E}(z,t) = \Re\left\{\bar{E}(z)e^{j\omega t}\right\} = \Re\left\{\hat{x}1000e^{-j\beta_0 z}e^{j\omega t}\right\} \text{ V/m}$$
$$\bar{H}(z,t) = \Re\left\{\bar{H}(z)e^{j\omega t}\right\} = \Re\left\{\hat{y}2.65e^{-j\beta_0 z}e^{j\omega t}\right\} \text{ A/m}$$

that is:

$$\bar{E}(z,t) = \hat{x}1000\cos(\omega t - \beta_0 z) \text{ V/m}$$
$$\bar{H}(z,t) = \hat{y}2.65\cos(\omega t - \beta_0 z) \text{ A/m}$$

Exercise 7 (cont.)

Furthermore we have:

$$\beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi(20 \cdot 10^6)}{(3 \cdot 10^8)} \simeq 0.42 \text{ m}^{-1}$$

$$v_p = \frac{\omega}{\beta_0} = c = 3 \cdot 10^8 \text{ m/s}$$

and

$$\lambda_0 = \frac{2\pi}{\beta_0} = \frac{c}{f} = \frac{3 \cdot 10^8}{20 \cdot 10^6} = 15 \text{ m}$$

Exercise 8

Determine the maximum angle of refraction and critical angle of reflection for

(a) an air-plexiglass interface and
(b) an air-water interface.

The refractive indices of plexiglass and water at RF frequencies are:
$n_{\text{plexi}} \simeq 1.5$ and $n_{\text{water}} \simeq 9$.
Solution:

There is really only one angle to determine, because if $n = 1$ and $n' = n'_{\text{plexi}}$, then $\sin(\vartheta_m) = n/n' = 1/n'_{\text{plexi}}$, and if $n = n_{\text{plexi}}$ and $n' = 1$, then, $\sin(\vartheta_c) = n'/n = 1/n_{\text{plexi}}$. Thus, $\vartheta_c = \vartheta_m$:

$$\vartheta_m = \arcsin \frac{1}{1.5} \simeq 0.73 \text{ rad} \simeq 41.8^\circ$$

For the air-water case, we have:

$$\vartheta_m = \arcsin \frac{1}{9} \simeq 0.11 \text{ rad} \simeq 6.38^\circ$$

Exercise 9

Given the vector field $\bar{V}(x, y, z)$:

$$\bar{V}(x, y, z) = (x + 3zy)\,\bar{x} + \left(z^2 - xy\right)\hat{y} + xy^2\hat{z}$$

determine $\nabla \cdot \bar{V}$ and $\nabla \times \bar{V}$.

Solution:

In Cartesian coordinates the divergence is:

$$\nabla \cdot \bar{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

then

$$\nabla \cdot \bar{V} = \frac{\partial\,(x + 3zy)}{\partial x} + \frac{\partial\left(z^2 - xy\right)}{\partial y} + \frac{\partial\left(xy^2\right)}{\partial z} = 1 - x$$

Exercise 9 (cont.)

The curl is, instead, is given by the symbolic determinant:

$$\nabla \times \bar{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} + $$
$$+ \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}$$

then

$$\nabla \times \bar{V} = \left(\frac{\partial\left(xy^2\right)}{\partial y} - \frac{\partial\left(z^2 - xy\right)}{\partial z}\right)\hat{x} +$$
$$+ \left(\frac{\partial\left(x + 3zy\right)}{\partial z} - \frac{\partial\left(xy^2\right)}{\partial x}\right)\hat{y} + \left(\frac{\partial\left(z^2 - xy\right)}{\partial x} - \frac{\partial\left(x + 3zy\right)}{\partial y}\right)\hat{z} =$$
$$= (2xy - 2z)\,\hat{x} + \left(3y - y^2\right)\hat{y} + (-y - 3z)\,\hat{z}$$

Exercise 10

A plane electromagnetic wave is normally incident from a free-space (medium 1) on a plane semi-infinite ideal dielectric (medium 2). The amplitude of the reflection coefficient at the interface is $|\Gamma| = 0.268$ and there is an average rate of flow of energy in the z-direction of 1 W/m^2.

Determine the relative permittivity of the dielectric, the maximum and minimum electric field intensities in medium 1, and the electric field intensity in medium 2.

Solution:

Assume

$$\bar{E}_1(\bar{r}) = \left(E_{1+}e^{-jk_0z} + \Gamma E_{1+}e^{jk_0z}\right)\hat{x}$$

$$\bar{H}_1(\bar{r}) = \frac{1}{\eta_0}\left(E_{1+}e^{-jk_0z} - \Gamma E_{1+}e^{jk_0z}\right)\hat{y}$$

$$\bar{E}_2(\bar{r}) = TE_{1+}e^{-jk_2z}\hat{x}$$

$$\bar{E}_2(\bar{r}) = \frac{1}{\eta_2}TE_{1+}e^{-jk_2z}\hat{y}$$

Exercise 10 (cont.)

Without loss of generality we can put the interface at $z = 0$. Then

$$E_{1+} + \Gamma E_{1+} = T E_{1+}$$
$$\frac{1}{\eta_0}\left(E_{1+} - \Gamma E_{1+}\right) = \frac{1}{\eta_2} T E_{1+}$$

and the amplitudes of the Poynting vectors are also equal:

$$\frac{1}{2}\frac{|E_{1+}|^2}{\eta_0}\left(1 - |\Gamma|^2\right) = \frac{1}{2}\frac{|E_{1+}|^2}{\eta_2}|T|^2 = 1\ \mathrm{W/m^2}$$

Furthermore, since medium 2 is a perfect dielectric, it will be:

$$\eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}} = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} < \eta_0 \qquad \eta_2 \in \mathbb{R} \qquad \Gamma = \frac{\eta_2 - \eta_0}{\eta_2 + \eta_0} = \frac{1 - \sqrt{\varepsilon_{r_2}}}{1 + \sqrt{\varepsilon_{r_2}}}$$

this means that:

$$\Gamma = -0.268$$

Exercise 10 (cont.)

and

$$-0.268\left(1+\sqrt{\varepsilon_{r_2}}\right)=1-\sqrt{\varepsilon_{r_2}} \qquad \sqrt{\varepsilon_{r_2}}\,(1-0.268)=1.268$$

$$\eta_2=\frac{\eta_0}{\sqrt{\varepsilon_{r_2}}}\simeq 217.64\,\Omega \qquad \varepsilon_{r_2}\simeq 3$$

From $T=1+\Gamma$ we have:

$$T=1-0.268=0.732$$

and from $\frac{1}{2}\frac{|E_{1+}|^2}{\eta_2}|T|^2=1$

$$|E_{1+}|^2=\frac{2\eta_2}{|T|^2} \qquad |E_{1+}|=\frac{\sqrt{2\eta_2}}{|T|}=\frac{\sqrt{2\cdot 271.64}}{0.732}\simeq 28.5\ \mathrm{V/m}$$

In general:

$$E_{1+}=|E_{1+}|\,e^{j\varphi}=28.5e^{j\varphi}\ \mathrm{V/m}$$

Exercise 10 (cont.)

The fields will be:

$$\bar{E}_1\left(\bar{r}\right) = \left(28.5\mathrm{e}^{-jk_0 z} - 0.268 \cdot 28.5\mathrm{e}^{jk_0 z}\right)\mathrm{e}^{j\varphi}\hat{x}$$

$$\bar{H}_1\left(\bar{r}\right) = \frac{1}{377}\left(28.5\mathrm{e}^{-jk_0 z} + 0.268 \cdot 28.5\mathrm{e}^{jk_0 z}\right)\mathrm{e}^{j\varphi}\hat{y}$$

$$\bar{E}_2\left(\bar{r}\right) = 0.732 \cdot 28.5\mathrm{e}^{j\varphi}\mathrm{e}^{-jk_2 z}\hat{x}$$

$$\bar{E}_2\left(\bar{r}\right) = \frac{1}{217.64}0.732 \cdot 28.5\mathrm{e}^{j\varphi}\mathrm{e}^{-jk_2 z}\hat{y}$$

It is clear that in medium 1 the field will be the sum of a progressive and of a standing wave.

There is no need to write explicitly the standing wave. Note, instead, that the field intensity will vary according to the phases of the incident and of the reflected terms.

Exercise 10 (cont.)

When:

$$e^{-jk_0 z} = e^{+jk_0 z}$$

the electric field intensity will be minimum (because $\Gamma < 0$) and:

$$|E_1|_{\text{min}} = 28.5\,(1 - 0.268) = 28.5 \cdot 0.732 \simeq 20.87 \text{ V/m}$$

Instead, when

$$e^{-jk_0 z} = -e^{+jk_0 z} = e^{j(2n+1)\pi} e^{+jk_0 z}$$

that this when incident and reflected fields are opposite in phase, we have a maximum of intensity for the electric field:

$$|E_1|_{\text{max}} = 28.5\,(1 + 0.268) = 28.5 \cdot 1.268 \simeq 36.14 \text{ V/m}$$

At the interface we have a minimum of intensity and, according to the interface conditions:

$$|E_2| = |E_1|_{\text{min}} = 28.5\,(1 - 0.268) = 28.5 \cdot 0.732 \simeq 20.87 \text{ V/m}$$

Exercise 11

We wish to shield a piece of equipment from RF interference over the frequency range from 10 kHz to 1 GHz by enclosing it in a copper enclosure. The conductibility of the copper is $5.8 \cdot 10^7$ Siemens/m. The RF interference inside the enclosure is required to be at least 180 dB down compared to its value outside. What is the minimum thickness t of the copper shield in mm?

Solution:

An exact solution, especially at the lowest frequencies would require the use of numerical codes. Nevertheless, the approximation of plane shield in far field (plane wave) can provide an acceptable result.

Since the shield is made of a good conductor:

$$k_{\text{copper}} = \beta - j\alpha = \frac{1}{\delta} - j\frac{1}{\delta}$$

where δ is the penetration depth.

Exercise 11 (cont.)

For copper $\mu = \mu_0$ and we have:

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{\cancel{2}}{\cancel{2}\pi f \mu\sigma}} = \frac{1}{\sqrt{f}} \frac{1}{\sqrt{\pi \cdot 4\pi \cdot \cancel{10^{-7}} 5.8 \cdot \cancel{10^{7}}}} \simeq \frac{0.0661}{\sqrt{f}} \text{ m}$$

Another important parameter is the intrinsic impedance:

$$\eta_{\text{copper}} = \sqrt{\frac{\omega\mu}{2\sigma}} (1+j) = \sqrt{\frac{\cancel{2}\pi f \mu}{\cancel{2}\sigma}} (1+j) =$$

$$= \sqrt{f} \sqrt{\frac{\pi \cdot 4\pi \cdot 10^{-7}}{5.8 \cdot 10^{7}}} (1+j) \simeq 2.60 \cdot 10^{-7} (1+j) \sqrt{f} \; \Omega$$

and its magnitude:

$$|\eta_{\text{copper}}| \simeq 2.60 \cdot 10^{-7} \sqrt{2} \sqrt{f} \; \Omega \simeq 3.677 \cdot 10^{-7} \sqrt{f} \; \Omega$$

Exercise 11 (cont.)

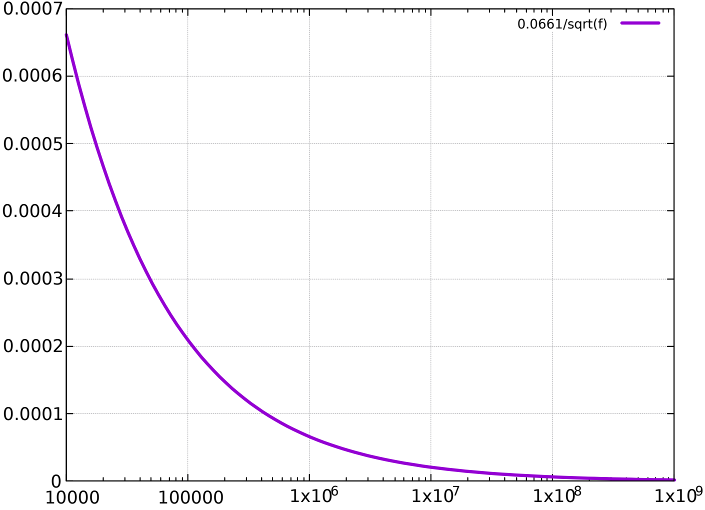


Figure: Skin depth for copper from 10 KHz to 1 GHz: δ ranges from $0.661 \cdot 10^{-3}$ to $2.09 \cdot 10^{-6}$

Exercise 11 (cont.)

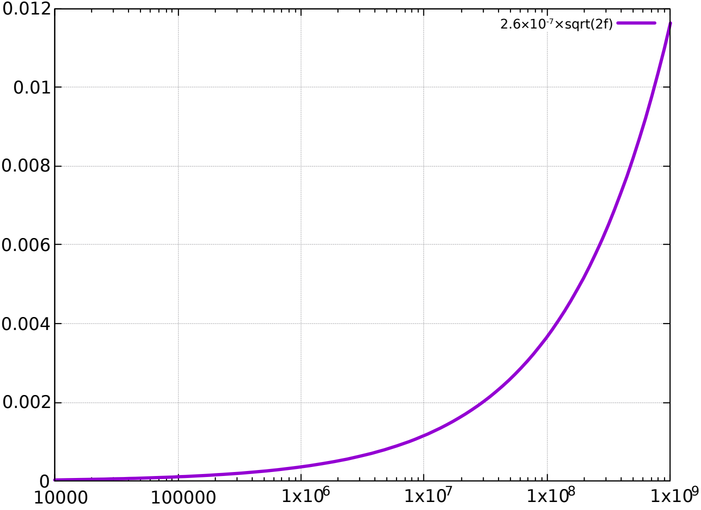


Figure: Magnitude of the intrinsic impedance of copper from 10 KHz to 1 GHz: η_{copper} ranges from $3.68 \cdot 10^{-5}$ to $1.16 \cdot 10^{-2}$

Exercise 11 (cont.)

The plane wave shielding effectiveness (in dB) can be expressed as the sum of a reflection loss R_{dB} and an absorption loss A_{dB} plus a multiple reflection correction term M_{dB}:

$$SE_{\mathrm{dB}} = R_{\mathrm{dB}} + A_{\mathrm{dB}} + M_{\mathrm{dB}}$$

where

$$R_{\mathrm{dB}} = 20\log_{10}\left|\frac{(\eta_{\mathrm{copper}} + \eta_0)^2}{4\eta_0\eta_{\mathrm{copper}}}\right|$$

$$A_{\mathrm{dB}} = 20\log_{10}\mathrm{e}^{\frac{t}{\delta}} = \frac{t}{\delta}20\log_{10}\mathrm{e} \simeq 8.6859\frac{t}{\delta}$$

and

$$M_{\mathrm{dB}} = 20\log_{10}\left|\left[1 - \left(\frac{\eta_{\mathrm{copper}} - \eta_0}{\eta_{\mathrm{copper}} + \eta_0}\right)^2 \mathrm{e}^{-2\frac{t}{\delta}}\mathrm{e}^{-j2\frac{t}{\delta}}\right]\right|$$

Exercise 11 (cont.)

Since $\eta_{\text{copper}} \ll \eta_0$ the reflection loss term can be approximated as:

$$R_{\text{dB}} = 20\log_{10}\left|\frac{(\eta_{\text{copper}}+\eta_0)^2}{4\eta_0\eta_{\text{copper}}}\right| \simeq 20\log_{10}\left|\frac{\eta_0}{4\eta_{\text{copper}}}\right|$$

By using this approximation we have[1]:

$$R_{\text{dB}_{10\text{ kHz}}} \simeq 128\text{ dB} \qquad R_{\text{dB}_{1\text{ GHz}}} \simeq 78\text{ dB}$$

then we need to add an absorption of at least:

$$A_{\text{dB}_{10\text{ kHz}}} = 52\text{ dB} \qquad A_{\text{dB}_{1\text{ GHz}}} \simeq 102\text{ dB}$$

[1] Note that the value of $R_{\text{dB}_{10\text{ kHz}}}$ is totally unrealistic, since $\lambda_{10\text{ kHz}} = 30$ Km and then it is unlikely that the source of the interference be in the far field of the equipment. This is accounted for in the exercise by asking for a shielding effectiveness of 180 dB, which is a very high value.

Exercise 11 (cont.)

Since the thickness of the shield is more critical at low frequencies, it is convenient to search fo a value of t satisfying the requirement at $f = 10$ kHz and successively verify if this value can be acceptable at $f = 1$ GHz. Since

$$A_{\mathrm{dB}} = \simeq 8.6859\frac{t}{\delta}$$

we require that:

$$A_{\mathrm{dB}_{10\ \mathrm{kHz}}} = 8.6859\frac{t}{\delta} = 8.6859\frac{t}{0.661 \cdot 10^{-3}} = 52$$

that is:

$$t = \frac{0.661 \cdot 10^{-3}}{8.6859}52 \simeq 4.0 \cdot 10^{-3}\ \mathrm{m} = 4.0\ \mathrm{mm}$$

If we use this value of t, at $f = 1$ GHz we have:

$$A_{\mathrm{dB}_{1\ \mathrm{GHz}}} = 8.6859\frac{4.0 \cdot 10^{-3}}{2.09 \cdot 10^{-6}} \simeq 17372\ \mathrm{dB!!!}$$

Exercise 11 (cont.)

Actually, to satisfy the requirements at $f = 1$ GHz it would have been sufficient a thickness t' such that:

$$8.6859 \frac{t'}{2.09 \cdot 10^{-6}} = 102$$

that is

$$t' = \frac{2.09 \cdot 10^{-6}}{8.6859} 102 \simeq 24.5 \cdot 10^{-6} \text{ m} = 24.5 \ \mu\text{m}$$

As for M_{dB}, it is bounded by:

$$0 \geq M_{\text{dB}} \geq 20 \log_{10} \left| 1 - \mathrm{e}^{-2\frac{t}{\delta}} \right|$$

This means that, at $f = 10$ kHz

$$M_{\text{dB}_{10\text{ kHz}}} \geq 20 \log_{10} \left| 1 - \mathrm{e}^{-2\frac{4.0 \cdot 10^{3}}{0.661 \cdot 10^{-3}}} \right| > -4.15 \cdot 10^{-5} \text{ dB} \simeq 0$$

therefore the effect of multiple reflections can be completely disregarded.

Exercise 12

Two semi-infinite media are separated by a plane interface at $z = 0$.

The following incident plane wave is propagating in medium 1 (vacuum):

$$\bar{E} = \left(-\frac{4}{3}\hat{x} - 2\hat{y} + \hat{z}\right) \mathrm{e}^{-j(3x+4z)}$$

$$\bar{H} = \frac{1}{\eta_0}\left(\frac{8}{5}\hat{x} - \frac{5}{3}\hat{y} - \frac{6}{5}\hat{z}\right) \mathrm{e}^{-j(3x+4z)}$$

Medium 2 is a perfect dielectric with $\varepsilon_{r_2} = 4$.

Determine the total fields in the two media.

Solution:

In accordance to the general form of a plane wave, it should be clear that:

$$\bar{k} \cdot \bar{r} = \bar{k} \cdot (x\hat{x} + y\hat{y} + z\hat{z}) = 3x + 0y + 4z$$

then

$$\bar{k} = 3\hat{x} + 4\hat{z} \qquad \left|\bar{k}\right| = k = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \qquad \hat{k} = \frac{\bar{k}}{k} = \frac{3}{5}\hat{x} + \frac{4}{5}\hat{z}$$

Exercise 12 (cont.)

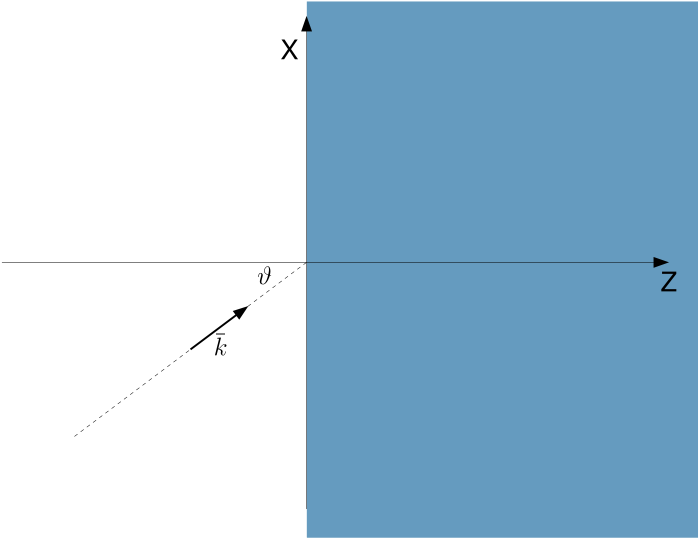


Figure: Schematic of the exercise (a)

Exercise 12 (cont.)

Now, we need to compute the wave vectors for both the reflected and the refracted wave.

Since

$$\bar{k} = k\left(\sin\vartheta\hat{x} + \cos\vartheta\hat{z}\right) = k\hat{k} = k\left(\frac{3}{5}\hat{x} + \frac{4}{5}\hat{z}\right)$$

$$\bar{k}'' = k''\left(\sin\vartheta''\hat{x} - \cos\vartheta''\hat{z}\right) = k''\hat{k}'' = k\hat{k}''$$

and, from the Snel laws

$$\vartheta = \vartheta''$$

we have, for the reflected wave:

$$\bar{k}'' = k\left(\frac{3}{5}\hat{x} - \frac{4}{5}\hat{z}\right) = 5\left(\frac{3}{5}\hat{x} - \frac{4}{5}\hat{z}\right) = 3\hat{x} - 4\hat{z}$$

Exercise 12 (cont.)

To compute the wave vector of the refracted wave we use

$$\bar{k}' = k' \left(\sin\vartheta' \hat{x} + \cos\vartheta' \hat{z}\right) = k' \hat{k}'$$

and the Snel law

$$k \sin\vartheta = k' \sin\vartheta'$$

By using the previous relations we find

$$\sin\vartheta' = \frac{k}{k'} \sin\vartheta = \frac{\cancel{\omega\sqrt{\mu_0\varepsilon_0}}}{\cancel{\omega\sqrt{\mu_0\varepsilon_0}}\sqrt{\varepsilon_{r_2}}} \sin\vartheta = \frac{1}{\sqrt{4}}\frac{3}{5} = \frac{3}{10}$$

$$\cos\vartheta' = \sqrt{1-\sin^2\vartheta'} = \sqrt{1-\frac{9}{100}} = \frac{\sqrt{91}}{10}$$

with $k' = k\sqrt{\varepsilon_{r_2}} = 5 \cdot 2 = 10$. Then:

$$\bar{k}' = k' \left(\sin\vartheta' \hat{x} + \cos\vartheta' \hat{z}\right) = k' \hat{k}' = 10 \left(\frac{3}{10}\hat{x} + \frac{\sqrt{91}}{10}\hat{z}\right) = 3\hat{x} + \sqrt{91}\hat{z}$$

Exercise 12 (cont.)

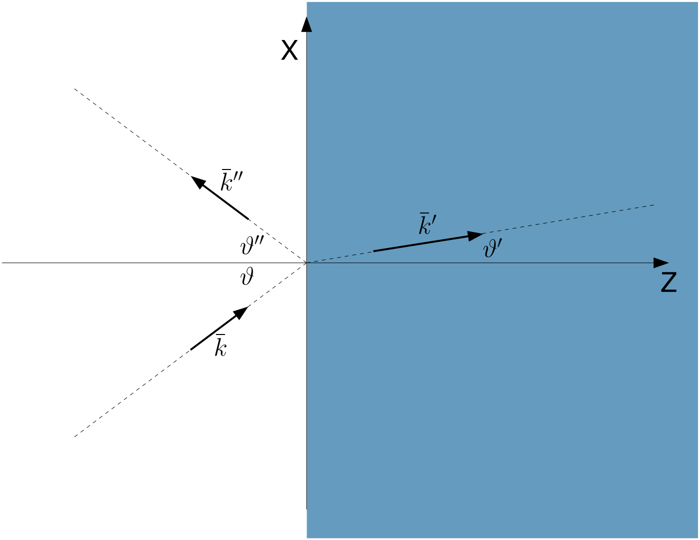


Figure: Schematic of the exercise (b)

Exercise 12 (cont.)

Next, we can compute the transverse impedances and decompose the fields into a *TM* and a *TE* part.

The transverse impedances of the incident and reflected waves in medium 1 are:

$$\zeta_{TM} = \eta_0 \cos\vartheta = \frac{4}{5}\eta_0 \qquad\qquad \zeta_{TE} = \frac{\eta_0}{\cos\vartheta} = \frac{5}{4}\eta_0$$

for the *TM* and the *TE* parts, respectively.

In medium 2, instead, we have:

$$\eta_2 = \eta' = \sqrt{\frac{\mu_0}{4\varepsilon_0}} = \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{2}\eta_0$$

$$\cos\vartheta' = \frac{\sqrt{91}}{10}$$

so that:

$$\zeta'_{TM} = \eta' \cos\vartheta' = \frac{1}{2}\eta_0\frac{\sqrt{91}}{10} = \frac{\sqrt{91}}{20}\eta_0 \quad \zeta'_{TE} = \frac{\eta'}{\cos\vartheta'} = \frac{1}{2}\eta_0\frac{10}{\sqrt{91}} = \frac{5}{\sqrt{91}}\eta_0$$

Exercise 12 (cont.)

Now, the Fresnel coefficients can be computed.

Since medium 2 is semi-infinite and source-free, we need to evaluate only the values "from the left to the right".

Since, in general

$$\rho_T = \frac{\zeta'_T - \zeta_T}{\zeta'_T + \zeta_T} \qquad \tau_T = \frac{2\zeta'_T}{\zeta'_T + \zeta_T}$$

then the TM coefficients are

$$\rho_{TM} = \frac{\zeta'_{TM} - \zeta_{TM}}{\zeta'_{TM} + \zeta_{TM}} = \frac{\frac{\sqrt{91}}{20}\cancel{\eta_0} - \frac{4}{5}\cancel{\eta_0}}{\frac{\sqrt{91}}{20}\cancel{\eta_0} + \frac{4}{5}\cancel{\eta_0}} = \frac{\sqrt{91} - 16}{\sqrt{91} + 16} \simeq -0.253$$

$$\tau_{TM} = \frac{2\zeta'_{TM}}{\zeta'_{TM} + \zeta_{TM}} = \frac{2\frac{\sqrt{91}}{20}\cancel{\eta_0}}{\frac{\sqrt{91}}{20}\cancel{\eta_0} + \frac{4}{5}\cancel{\eta_0}} = \frac{2\sqrt{91}}{\sqrt{91} + 16} \simeq 0.747$$

Exercise 12 (cont.)

The *TE* coefficients are:

$$\rho_{TE} = \frac{\zeta'_{TE} - \zeta_{TE}}{\zeta'_{TE} + \zeta_{TE}} = \frac{\frac{\cancel{5}}{\sqrt{91}}\cancel{\eta_0} - \frac{\cancel{5}}{4}\cancel{\eta_0}}{\frac{\cancel{5}}{\sqrt{91}}\cancel{\eta_0} + \frac{\cancel{5}}{4}\cancel{\eta_0}} = \frac{4 - \sqrt{91}}{4 + \sqrt{91}} \simeq -0.409$$

$$\tau_{TE} = \frac{2\zeta'_{TE}}{\zeta'_{TE} + \zeta_{TE}} = \frac{2\frac{\cancel{5}}{\sqrt{91}}\cancel{\eta_0}}{\frac{\cancel{5}}{\sqrt{91}}\cancel{\eta_0} + \frac{\cancel{5}}{4}\cancel{\eta_0}} = \frac{8}{4 + \sqrt{91}} \simeq 0.591$$

Exercise 12 (cont.)

As for the decomposition of the fields into a *TM* and a *TE* parts, remember that the incident fields can be decomposed as:

$$\bar{E}(\bar{r}) = \bar{E}_+ e^{-j\bar{k}\cdot\bar{r}} = \left[\left(\hat{x}\cos\vartheta - \hat{z}\sin\vartheta\right)A_+ + \hat{y}B_+\right] e^{-jk(x\sin\vartheta + z\cos\vartheta)}$$

$$\begin{aligned}\bar{H}(\bar{r}) &= \bar{H}_+ e^{-j\bar{k}\cdot\bar{r}} = \hat{k}\times\frac{\bar{E}_+}{\eta}e^{-j\bar{k}\cdot\bar{r}} = \\ &= \frac{1}{\eta}\left[\hat{y}A_+ - \left(\hat{x}\cos\vartheta - \hat{z}\sin\vartheta\right)B_+\right] e^{-jk(x\sin\vartheta + z\cos\vartheta)}\end{aligned}$$

with

$$\begin{aligned}\bar{E}_+ &= \left(\hat{x}\cos\vartheta - \hat{z}\sin\vartheta\right)A_+ + \hat{y}B_+ \\ \bar{H}_+ &= \frac{1}{\eta}\left[\hat{y}A_+ - \left(\hat{x}\cos\vartheta - \hat{z}\sin\vartheta\right)B_+\right]\end{aligned}$$

Similar relations hold for the reflected and the refracted fields.

Exercise 12 (cont.)

So we need to satisfy

$$\left(-\frac{4}{3}\hat{x} - 2\hat{y} + \hat{z}\right) = (\hat{x}\cos\vartheta - \hat{z}\sin\vartheta)\, A_+ + \hat{y}B_+ =$$

$$= \left(\hat{x}\frac{4}{5} - \hat{z}\frac{3}{5}\right) A_+ + \hat{y}B_+$$

$$\frac{1}{\eta_0}\left(\frac{8}{5}\hat{x} - \frac{5}{3}\hat{y} - \frac{6}{5}\hat{z}\right) = \frac{1}{\eta_0}\left[\hat{y}A_+ - (\hat{x}\cos\vartheta - \hat{z}\sin\vartheta)\, B_+\right] =$$

$$= \frac{1}{\eta_0}\left[\hat{y}A_+ - \left(\hat{x}\frac{4}{5} - \hat{z}\frac{3}{5}\right) B_+\right]$$

that is:

$$A_+ = -\frac{5}{3} \quad \text{(TM wave)} \qquad\qquad B_+ = -2 \quad \text{(TE wave)}$$

The transverse components are:

$$A_{T+} = A_+\cos\vartheta = -\frac{5}{3}\cdot\frac{4}{5} = -\frac{4}{3} \qquad\qquad B_{T+} = B_+ = -2$$

Exercise 12 (cont.)

Since

$$\rho_{TM} = \frac{A_- \cos\vartheta}{A_+ \cos\vartheta} = \frac{A_-}{A_+} \qquad \rho_{TE} = \frac{B_-}{B_+}$$

and

$$\tau_{TM} = \frac{A'_+ \cos\vartheta'}{A_+ \cos\vartheta} = \frac{\cos\vartheta'}{\cos\vartheta}\,\frac{A'_+}{A_+} \qquad \tau_{TE} = \frac{B'_+}{B_+}$$

we have:

$$A_- = \rho_{TM} A_+ = -0.253 \cdot \left(-\frac{5}{3}\right) \qquad B_- = \rho_{TE} B_+ = -2 \cdot (-0.409)$$

and

$$A'_+ = \tau_{TM} A_+ \frac{\cos\vartheta}{\cos\vartheta'} = 0.747 \cdot \left(-\frac{5}{3}\right) \frac{\frac{4}{5}}{\frac{\sqrt{91}}{10}} \quad B'_+ = \tau_{TE} B_+ = -2 \cdot 0.591$$

Exercise 12 (cont.)

To summarize:

$$A_+ \simeq -1.667 \qquad A_- \simeq 0.422 \qquad A'_+ \simeq 1.044$$
$$B_+ = -2 \qquad B_- = 0.818 \qquad B'_+ = -1.182$$

and

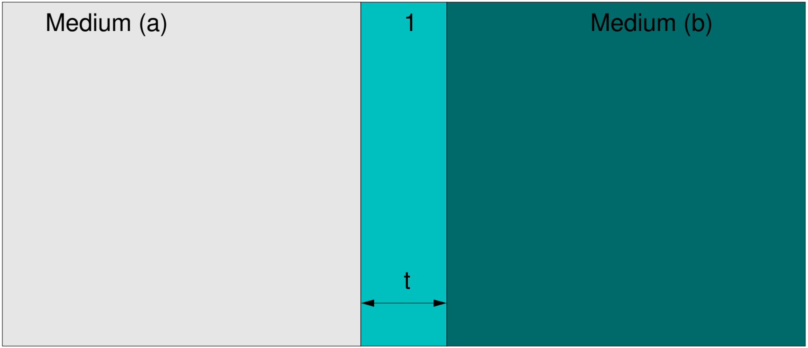
$$\cos\vartheta = \cos\vartheta'' = 0.8 \qquad \sin\vartheta = \sin\vartheta'' = 0.6$$
$$\cos\vartheta' \simeq 0.954 \qquad \sin\vartheta' = 0.3$$

$$\eta_1 = \eta_0 \qquad \eta_2 = \eta' = 0.5\eta_0$$
$$n_1 = n_0 = 1 \qquad n_2 = n' = 2$$

By means of these values we can write the total fields in medium 1 and in medium 2. This would take some slides and is omitted[2].

[2] Formulas are in the Orfanidis book pp.245-246, eqs. (7.2.17–7.2.20)

Exercise 13



Determine the dielectric constant ε_{r_1} and the thickness t of a quarter-wave anti-reflection slab to match two semi-infinite media: medium (a) is air, medium (b) is a perfect dielectric with $\varepsilon_{r_b} = 9$. The working frequency is $f = 3$ GHz.

Exercise 13 (cont.)

Solution:

In order to avoid reflections we interpose a dielectric layer of characteristic impedance η_1, such that:

$$\eta_1^2 = \eta_a \eta_b$$

Since

$$\eta_1^2 = \left(\frac{1}{\sqrt{\varepsilon_{r_1}}}\eta_0\right)^2 = \frac{1}{\varepsilon_{r_1}}\eta_0^2 \qquad \eta_a\eta_b = \eta_0 \frac{1}{\sqrt{\varepsilon_{r_b}}}\eta_0 = \frac{1}{3}\eta_0^2$$

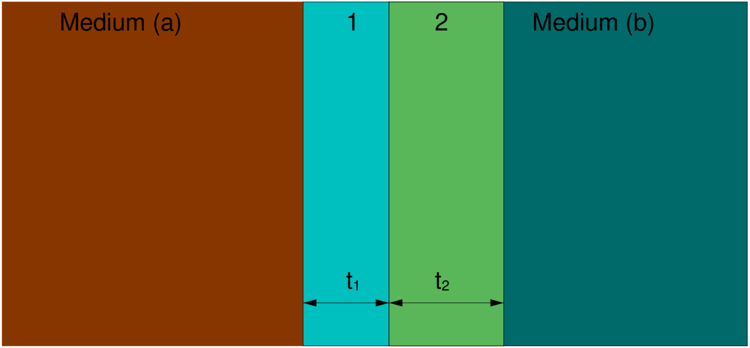
we have $\varepsilon_{r_1} = 3$.

Since the wavelength in the slab is:

$$\lambda_1 = \frac{c_0}{f}\frac{1}{\sqrt{\varepsilon_{r_1}}} \simeq 5.77 \cdot 10^{-2} \text{ m} = 5.77 \text{ cm}$$

the thickness must be $t \simeq 1.44$ cm.

Exercise 14



We want to avoid reflections when a plane wave with normal incidence is propagated from the (semi-infinite) medium (a) to the (semi-infinite) medium (b). To match the two media, two slabs, 1 and 2, whose thickness are t_1 and t_2, respectively, are used. Medium (a) has a relative dielectric permittivity $\varepsilon_{r_a} = 14.44$, while in medium (b) we have $\varepsilon_{r_b} = 3.61$.

Exercise 14 (cont.)

- Design the slabs (define suitable values of t_1 and t_2 and of the respective dielectric permittivities) to have a perfect match at the frequency $f = 1$ GHz. (Assume $\mu_a = \mu_1 = \mu_2 = \mu_b = \mu_0$).
- Is the solution unique?

Solution:

Since we use two slabs for matching, there are many pairs $(t_1, \eta_1), (t_2, \eta_2)$ that can provide matching between η_a and η_b, hence the solution is not unique.

With arbitrary t_1 and t_2, the general solution is expressed by a somewhat tricky formula. In order to simplify things, it is customary to assume that each matching layer be a quarter-wavelength in length. In the latter case, the reflection coefficient between medium (a) and medium (1) becomes[3]:

$$\Gamma_a = \frac{\eta_1^2 \eta_b - \eta_2^2 \eta_a}{\eta_1^2 \eta_b + \eta_2^2 \eta_a}$$

[3] Orfanidis, p. 177, Example 5.7.1

Exercise 14 (cont.)

In matching conditions, $\Gamma_a = 0$, then

$$\eta_1^2\eta_b - \eta_2^2\eta_a = 0 \qquad \eta_1^2\eta_b = \eta_2^2\eta_a$$

and, since

$$\eta_a = \eta_0 \frac{1}{\sqrt{\varepsilon_{r_a}}} \qquad \eta_1 = \eta_0 \frac{1}{\sqrt{\varepsilon_{r_1}}} \qquad \eta_2 = \eta_0 \frac{1}{\sqrt{\varepsilon_{r_2}}} \qquad \eta_b = \eta_0 \frac{1}{\sqrt{\varepsilon_{r_b}}}$$

we need

$$\frac{1}{\varepsilon_{r_1}\sqrt{\varepsilon_{r_b}}} = \frac{1}{\varepsilon_{r_2}\sqrt{\varepsilon_{r_a}}} \qquad \frac{\varepsilon_{r_1}}{\varepsilon_{r_2}} = \frac{\sqrt{\varepsilon_{r_a}}}{\sqrt{\varepsilon_{r_b}}} \qquad \varepsilon_{r_1} = \varepsilon_{r_2}\frac{\sqrt{\varepsilon_{r_a}}}{\sqrt{\varepsilon_{r_b}}}.$$

Again, we see from the last formula, that the solution is not unique and that any pair

$$\left(\varepsilon_{r_2}\frac{\sqrt{\varepsilon_{r_a}}}{\sqrt{\varepsilon_{r_b}}}, \varepsilon_{r_2}\right) \text{ or } \left(\varepsilon_{r_1}, \varepsilon_{r_1}\frac{\sqrt{\varepsilon_{r_b}}}{\sqrt{\varepsilon_a}}\right)$$

with arbitrary ε_{r_2} (resp. ε_{r_1}) can be used.

Exercise 14 (cont.)

With values given in the exercise, we have

$$\frac{\sqrt{\varepsilon_{r_a}}}{\sqrt{\varepsilon_{r_b}}} = \frac{\sqrt{14.44}}{\sqrt{3.61}} = \frac{3.8}{1.9} = 2$$

and we could, for example, choose:

$$\varepsilon_{r_1} = 9 \qquad \varepsilon_{r_2} = 4.5$$

Since the frequency is $f = 1$ GHz, then

$$\lambda_0 = \frac{c_0}{f} = \frac{3 \times 10^8}{1 \times 10^9} = 0.3 \text{ m}$$

so we have

$$\lambda_1 = \frac{\lambda_0}{\sqrt{\varepsilon_{r_1}}} = \frac{0.3}{\sqrt{9}} = \frac{0.3}{3} = 0.1 \quad \lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_{r_2}}} = \frac{0.3}{\sqrt{4.5}} \approx \frac{0.3}{2.12} \approx 0.141$$

Exercise 14 (cont.)

Since we used quarter-wavelength slabs, the thicknesses are

$$t_1 = \frac{\lambda_1}{4} = \frac{0.1}{4} = 0.025 \text{ m} = 2.5 \text{ cm}$$

$$t_2 = \frac{\lambda_2}{4} = \frac{0.141}{4} \approx 0.035 \text{ m} = 3.5 \text{ cm}$$

respectively.

Note that, while at the design frequency f the choice of ε_{r_2} (resp. ε_{r_1}) does not matter, since we have $\Gamma_a(f) = 0$ in any case, the behavior of Γ_a can notably be affected in a band of frequencies, especially for broad bandwidths, depending on the choice of $(\varepsilon_{r_1}, \varepsilon_{r_2})$.

Exercise 15

A uniform plane wave propagating in vacuum along the z–direction has real–valued electric field components:

$$E_x = 4\cos(\omega t - kz + \frac{\pi}{4}), E_y = 2\sin(\omega t - kz - 0.9).$$

Its phasor form has the form $\bar{E} = (A\hat{x} + B\hat{y})e^{\pm jkz}$.

- Determine the numerical values of the complex–valued coefficients A, B and the correct sign of the exponent.
- Verify the result.
- Determine the polarization of this wave.

Solution

A general solution to this kind of problem can be achieved by observing that the real part of the quantity

$$\bar{E}e^{j\omega t}$$

must give the real vector in time domain.

Exercise 15 (cont.)

First of all, it is worth expressing the coefficients A and B in polar form:

$$A = |A|\, e^{j\varphi_A} \qquad\qquad B = |B|\, e^{j\varphi_B}$$

so that:

$$\bar{E} e^{j\omega t} = (A\hat{x} + B\hat{y})\, e^{\pm jkz} e^{j\omega t} = \left(|A|\, e^{j\varphi_A}\hat{x} + |B|\, e^{j\varphi_B}\hat{y}\right) e^{\pm jkz} e^{j\omega t}$$

and

$$\Re\left\{\bar{E} e^{j\omega t}\right\} = \Re\left\{|A|\, e^{\pm jkz} e^{j\omega t} e^{j\varphi_A}\hat{x}\right\} + \Re\left\{|B|\, e^{\pm jkz} e^{j\omega t} e^{j\varphi_B}\hat{y}\right\}$$

Since $|A|$, $|B|$, $\hat{x}$, and $\hat{y}$ are real-valued quantities, we have:

$$\Re\left\{|A|\, e^{\pm jkz} e^{j\omega t} e^{j\varphi_A}\hat{x}\right\} = |A|\, \Re\left\{e^{\pm jkz} e^{j\omega t} e^{j\varphi_A}\right\} \hat{x} = $$
$$= |A| \cos\left(\omega t \pm kz + \varphi_A\right) \hat{x}$$

$$\Re\left\{|B|\, e^{\pm jkz} e^{j\omega t} e^{j\varphi_B}\hat{y}\right\} = |B|\, \Re\left\{e^{\pm jkz} e^{j\omega t} e^{j\varphi_B}\right\} \hat{y} = $$
$$= |B| \cos\left(\omega t \pm kz + \varphi_B\right) \hat{y}$$

Exercise 15 (cont.)

By using the first result and equating the $\hat{x}$ components, we have:

$$|A| \cos(\omega t \pm kz + \varphi_A) = 4\cos(\omega t - kz + \frac{\pi}{4})$$

and, by inspection, we get immediately

$$A = |A| e^{j\varphi_A} = 4e^{j\pi/4}.$$

Furthermore we observe that the right sign of kz must be the minus sign.

As for the $\hat{y}$ component, the previous expansion imply:

$$|B| \cos(\omega t \pm kz + \varphi_B) = 2\sin(\omega t - kz - 0.9)$$

Exercise 15 (cont.)

There are many slightly different ways to solve the last equation. For example, remember the trigonometric identity:

$$\cos(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right)$$

then

$$|B|\cos(\omega t \pm kz + \varphi_B) = |B|\sin\left(\omega t \pm kz + \varphi_B + \frac{\pi}{2}\right)$$

and

$$|B| = 2 \qquad \varphi_B + \frac{\pi}{2} = -0.9 \qquad \varphi_B = -0.9 - \frac{\pi}{2}$$

therefore

$$B = 2e^{j(-0.9-\pi/2)} = 2e^{-j\pi/2}e^{-j0.9} = -j2e^{-j0.9}$$

Exercise 15 (cont.)

Finally, we have:

$$\bar{E} = (4e^{j\pi/4}\hat{x} - j2e^{-j0.9}\hat{y})e^{-jkz}$$

To verify our result we can multiply by $e^{j\omega t}$, expand the complex exponential terms according to Euler's formula, and take the real part.

$$\begin{aligned} \bar{E}e^{j\omega t} &= 4\left[\cos(\omega t - kz + \frac{\pi}{4}) + j\sin(\omega t - kz + \frac{\pi}{4})\right]\hat{x} + \\ &\quad - j2\left[\cos(\omega t - kz - 0.9) + j\sin(\omega t - kz - 0.9)\right]\hat{y} = \\ &= \left[4\cos(\omega t - kz + \frac{\pi}{4})\hat{x} + 2\sin(\omega t - kz - 0.9)\hat{y}\right] + \\ &\quad + j\left[4\sin(\omega t - kz + \frac{\pi}{4})\hat{x} - 2\cos(\omega t - kz - 0.9)\hat{y}\right] \end{aligned}$$

Exercise 15 (cont.)

The real part therefore is:

$$\Re\left\{\bar{E}e^{j\omega t}\right\} = 4\cos(\omega t - kz + \frac{\pi}{4})\hat{x} + 2\sin(\omega t - kz - 0.9)\hat{y}$$

and our solution is verified.

For the polarization:

since the x and y components are not in-phase we can exclude linear polarization;

furthermore, since the x and y components are not in quadrature (and also they have different amplitudes) polarization is not circular,

hence the polarization of the wave is elliptical.